CAD PACKAGE FOR THE DESIGN OF LQ OUTPUT FEEDBACK REGULATORS

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CERTIFICATE

It is certified that the present work entitled 'CAD Package for the Design of LQ Output Feedback Regulators' has been carried out by Mr. Babu Narayanan K.J. under my supervision and that it has not been submitted elsewhere for a degree.

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- Babu Narayanan, K.J.

ABSTRACT

A CAD package for the design of linear quadratic regulators using output feedback is developed. Various existing methods for the design were studied and an efficient algorithm, incorporating better features of some of them, was developed using linear static and dynamic output feedback. Several numerical examples have been worked out using the package developed. The user's guide for the use of the package developed is also given.

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Notations

$\lvert \cdot \rvert$	determinant of a matrix valued function or the absolute value of a scalar function
C	field of complex numbers
$\bar{C}$	field of complex numbers with real part < 0
$C^{n \times m}$	matrices with n rows and m columns with elements in C
R	field of real numbers
$R^{n \times m}$	matrices with n rows and m columns with elements in R
$\leq$, $\geq$	usual meaning for scalar and element by element application to matrices
$\lVert \cdot \rVert_p$	p-norm
$A > 0$	positive definite A
$A \geq 0$	non-negative definite A
A'	transpose of A
$Tr[\cdot]$	trace of a matrix
$E[\ \ ]$	expected value
$\partial J/\partial A$	gradient of scalar J with respect to matrix A
$\rho(A)$	spectrum of A
$\underline{x}$	a vector consisting of elements $x_1, x_2, \ldots$

$Col(A) = \underline{a}: = \underline{a}' = [\underline{A}_1', \underline{A}_2', \ldots, \underline{A}_n']$

A_i - ith column of A $R^{n \times n}$

CHAPTER 1

INTRODUCTION

The cornerstone of modern control theory judging from frequency of applications is the linear constant gain feedback control determined by minimizing a quadratic index of performance. With this LQR design approach, which is one of the most powerful state space methods of designing multivariable systems, design problems are reduced to choosing weighting matrices in the quadratic performance index. Moreover, the resulting closed loop system has good robustness properties.

But the implementation of LQR regulators require availability of all states of the system for measurement and feedback which, however, will be rare in practical situations. When the complete set of states are not available, in the case of stochastic systems a Kalman filter and in the case of deterministic system a Luenberger observer could be used to estimate the states and design an optimal controller. But the order of Kalman filter is equal to that of the system making the use of it for high order systems prohibitive. Although a full order observer too has an order the same as that of the system, reduced order observers and minimal order

or function observers exist to estimate the states. But even in these cases, the order may be too high for economical implementation.

Industrial processes are generally of high order and use of observers and optimal state control will be too sophisticated when often a simple controller that would stabilize the process in some operating region and achieve some desired behaviour would suffice. In such cases memoryless or state output feedback using only the available measurements will be suitable, making the structure of feedback simple.

The index of performance of optimal output feedback regulator is of course worse than that of the standard state feedback regulator, but this is the price paid for simplicity. In many cases, however, the performance of optimal output feedback regulators is close to that of standard regulator, thus justifying the use of simpler feedback structures.

There will be cases when there exists no static output feedback which can optimize a performance index. This happens when the system is not output stabilizable. Then a low order dynamic controllers is designed such that the overall closed loop system has satisfactory and desirable response characteristics. Almost all systems can be stabilized by dynamic output feedback, when they are controllable and observable [11]. No explicit estimation of states is done in dynamic feedback

controllers, which uses only available outputs for feedback. Since these methods do not require input sensing, unlike an observer, these have wider applications, especially when system inputs are unknown. Since observers are not unique selection of a particular observer is not easy.

In many applications the structure of feedback will have to be constrained, wherein some of the feedback elements are zeroes. One such case is that of a decentralized control of a geographically distributed system in which exchange of information can be prohibitively costly. It further simplifies the feedback structure and helps in eliminating unproductive elements in the feedback structure.

Unlike finding an optimal state feedback, which involves only the solution of an associated algebraic Riccati equation, the problem of finding an optimal output feedback is nontrivial and encounters many numerical problems.

The problem has been studied in detail and different solutions have come out, considering the cost and availability of computing system etc. Finer aspects of the problem like minimizing the order of compensator, developing numerically stable and efficient algorithms have taken significance.

In the work presented here, the algorithm mainly relied upon is a modified version of Anderson Moore method [42] which is based on a set of necessary conditions derived by Levine and Athans [39].

The thesis is organised as follows : Chapter 2 gives a brief survey of the literature in the area of output feedback regulators; Chapter 3 presents the static output feedback problem, algorithms used and briefly describes the priming feedback problem, search direction and unidimensional search along this direction; Chapter 4 describes the dynamic output feedback problem and how it is transformed into an equivalent static output feedback regulator problem; Chapter 6 analyses and discusses some of the results and Chapter 5 gives guidelines to the user for the use of the design package developed with the algorithm. Chapter 7 gives the conclusions.

CHAPTER 2

LITERATURE SURVEY

Linear quadratic optimal output feedback control has continuously stimulated research in the area of output feedback control, due to simplicity of constant feedback structure, ease in extending to dynamic output feedback and challenge in numerical solution for computing optimal gains.

The problem has been considered by several authors, but being a nontrivial computational problem, it has not yet found a fully acceptable solution.

Significant developments in the direction of finding optimal control with only accessible states were made in the early 1970's. However, progress in the direction of finding efficient algorithms for implementation were made much later. One of the problems in the design of optimal output feedback was and still is that of the choice of a performance index the minimization of which will result in a regulator as close in terms of the index as possible to the standard optimal regulator. Another problem associated with the design is the undesirable dependence of the feedback gain elements on the initial states, $\underline{x}_0$.

Minimization of maximum of the eigenvalues of the associated cost matrix, S, was done by Man [43]. He showed that this method is suprior, i.e., cost is always smaller, than minimizing the norm of the deviation of this matrix from the standard optimal cost matrix. An equivalent method was presented by Rekasius [53] in which, maximum, for any initial state, of the ratio of suboptimal cost to optimal cost was minimized. An upper bound of the above cost, Euclidean norm of cost matrix,was minimised in [18]. Levine and Athans [39] took the same performance index as in standard regulator and obtained necessary conditions for the solution. The dependancy of the solution on the initial state was eliminated by assuming the initial state to be a random vector uniformly distributed over the unit hypersphere and minimizing the expected value of the cost. A general performance index, (1), based on eigenvalues of S was minimized in [63] and [3] .

$$J = [\sum_{i=1}^{n} \lambda_i^k]^{1/k} = [\mathrm{Tr}\ (S^k)]^{1/k} \qquad (1)$$

The worst case of the above, when $k \longrightarrow \infty$, is the maximum eigenvalue of S. The sequence of J_k's, $k = 1,2,...$ are minimized in [3]. For $k = 1$, $J = \mathrm{Tr}\ [S]$, and the above problem reduces to the same as the one presented in [39].

Allwright considered a traceless formulation of performance criterion in [2] and arrived at a Riccati-like differential equation, the solution of which leads to a closed loop system which will never be worse than the open loop system for any initial state, x_o. In [1] Allwright found a sequence of feedback gain matrices F_k such that the performance of any F_k is better than the previous one, F_{k-1}, for any initial state x_o.

Bengtsson et al. [9] adopted a method for design of systems with inaccessible states in which a dominant eigenspace of the closed loop system with state feedback design is tried to be retained as the feedback structure is reduced from state feedback structure to required constrained output feedback structure. More recently Medanic et al. [45] used projection techniques to retain a dominant invariant subspace of the corresponding standard regulator in the output feedback solution. This method gives more insight into the properties of the corresponding closed loop regulators. [44] extends the method to discrete case.

A transfer function approach in which the optimal systems transfer function is approximated by output feedback transfer function was presented by Therapos [57]. Eitelberg et al. [20] showed that optimal state feedback gains of a suitably reduced order model is almost optimal, as output feedback gains to the original system.

Much of the work in the area of output feedback regulators are based on the modelling of initial states x_o as a random vector along the lines of the seminal paper by Levine and Athans [39].

Different methods were tried to solve the output feedback regulator problem presented in [39] by several authors. Many took to general purpose gradient methods for the parameter optimization. Davidon-Fletcher-Powell method (DFP method) was used in [16] and [28]; steepest descent method was used in [27]; conjugate gradient method was employed in [62]; and Broyden-Fletcher-Golub-Shanno method (BFGS method) was tried in [35]. Newton's method based on second order Taylor series expansion of the loss function was applied to the problem in [60], with conjugate gradient method.

The basic Levine-Athans algorithm, i.e.,solution of nonlinear equations, and its dual were presented with their global convergence properties in [58]. Anderson and Moore [4] used linear set of equations from the necessary conditions to find an optimal output feedback. The original Anderson-Moore algorithm was, however, found to diverge, as shown in [56].

Modified versions of the algorithm were presented by Moerder et al. [48] and Makila [41] and convergence of the methods with mild assumptions were proved. Several methods

for finding optimal constant output feedback gains and their connections to loss function expansion are discussed in [61].

O'Reilly [50] considers discrete time deterministic systems. Discrete-time stochastic systems are considered in [22], [41] and [61] and continuous time stochastic systems in [37].

It was shown by Kuhn et al. [35] that a deterministic problem with initial states and disturbances at the output modelled as random vectors and stochastic problems lead to the same optimal output feedback law.

One major disadvantage of all the algorithms is that they require a stabilizing feedback as the initial guess. This problem was circumvented by formulating an unconstrained optimization problem formed by augmenting the quadratic performance index with the structure constraints as an equality constraint by Choi et al. [17], Wenk and Knapp [62] and Shapiro et al. [55]. Bingulac et al. [10] solved a sequence of auxiliary optimal problems starting with full state feedback to get the solution. The above methods assumes that a stabilizing state feedback is available. With the available methods, finding a stabilizing state feedback is not a difficult problem.

In many practical situations, the feedback elements may be constrained to be zeroes or linearly related to other

elements. Such constrained structure was applied to flight control systems in [55]; problems in robotics in [62]; to large systems, in which costs of inter subsystem communications are significant, in [40]; in which subsystems are weakly coupled in [52] and to decentralized system in [59]. Necessary conditions for solution of linear quadratic problems with multiple structure constraints on state feedback, which extend directly to output feedback cases, were developed in [34]. Arkun et al. [5] gave lower and upper bounds of structure constrained controller costs which can be used for comparative assessment of different structures.

Kurtaran et al. [36] and El-Sayed [21] gave methods to choose an optimal pair of observation matrix and feedback matrix for a fixed number of outputs, for the cases when observation matrix, C, is free to be chosen. A lower bound of cost for any l-output system is thus found in [36]. Optimization of sensor locations subject to constraints is done in [21].

Much of the studies in output feedback problem has been done with infinite time problems and very little with finite time problems. Finite time regulators for continuous time stochastic system are studied in [37] and for discrete time stochastic case in [22], with the feedback gains varying with time.

In [13], regulation is done in finite time using a class of controls constrained to be the set of linear functions of outputs sampled at discrete instants of time and the feedback gains computed on-line.

For static feedback gains solution to infinite time problem may not exist. Mullis [49] showed that periodically varying gains always exist so as to stabilize the system.

When static feedback gains fail to find a solution to the problem, compensators are added till a satisfactory performance is attained. The results of static feedback case were extended to dynamic output feedback case by Johnson et al. [31] by augmenting the system states with compensator states. The procedure was extended to different types of systems by several authors [47]. Efficient algorithms to find an optimal dynamic output feedback compensator of fixed order based on the above procedure are given in [35] and [62].

To translate performance requirements like overshoot, settling time etc. time multiplied performance indices were used in [64] for continuous time case and in [14] and [38] for discrete time case.

In [15] theory of optimal output feedback control is extended to a class of quadratic performance measures whose weighting matrices are functions of frequency.

Since the existence of a solution to the optimal output feedback problem depends upon existence of a stabilizing feedback, results in output feedback stabilization like the one in [12] are of much interest.

Current research in the area of optimal output feedback are robust output feedback control and optimal controls for two-time scale systems [32].

CHAPTER 3

STATIC OUTPUT FEEDBACK

3.1 INTRODUCTION

As discussed in the previous chapter, the static output feedback problem has been studied in great detail resulting in many methods for the solution to the problem considering cost, availability of computing system etc. With the different methods having their own advantages and disadvantages. The epithet 'existing most efficient algorithm' does not fit any method. However, some of the methods have found wider application owing to factors like simplicity, amenability to use of modern digital computers, etc.

The algorithm in this thesis is similar to modified descent Anderson Moore algorithm (also called descent mapping algorithm) used in [41] [48]. The general output feedback problem, with system having direct feedthrough, quadratic cost having cross weighting of state and control and feedback gain matrix to be of constrained structure is considered here.

3.2 OUTPUT FEEDBACK REGULATOR PROBLEM

3.2.1 Continuous time systems

Consider the linear time-invariant continuous time system,

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{1}$$

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \tag{2}$$

$$\underline{y}_p(t) = C_p\underline{x}(t) + D_p\underline{u}(t) \tag{3}$$

where

$\underline{x} \in R^n$ is the system vector, $\underline{u} \in R^m$ the control inputs, $\underline{y} \in R^1$, measured outputs and $\underline{y}_p \in R^{1_p}$ outputs to be controlled. A, B, C, C_p, D and D_p are real constant matrices of compatible dimensions.

Realization (A, B, C) is assumed to be complete (complete controllability and complete observability) with rank $[C] = 1$ and (A, B, C_p) is assumed to be complete with rank $[C_p] = 1_p$.

The system, (1) - (3), is to be controlled by means of constant output feedback. Thus, $\underline{u}(t)$ is to be realised as

$$\underline{u}(t) = -F\,\underline{y}(t) \tag{4}$$

where $F \in R^{m\times 1}$ such that the index of performance or cost function

$$J(F) = \int_0^\infty (y_p' Q_1 y_p + 2u'T_1 y_p + u'Ru)dt + \gamma(F) \tag{5}$$

is minimised.

$\gamma(F)$ is a scalar penalty function on feedback matrix elements having a continuous gradient with respect to F and for which J is bounded below for all Fs which render

closed loop dynamics asymptotically stable, and Q_1, R_1 and T_1 are matrices satisfying the conditions, $Q_1 = Q_1' \geq 0$, $R_1 = R_1' > 0$, and $Q_1 - T_1' R_1^{-1} T_1 > 0$.

The minimization of the objective function is performed in the set S_F of stabilizing feedback gains

$$S_F = (F \in R^{m \times l} \mid \rho(A-BFC) \in C^-) \qquad (6)$$

where

$\rho(\cdot)$ denotes the spectrum, and C^- denotes open left-half complex plane.

A necessary condition for existance of solution to the output feedback problem defined above is that the minimizing F^* should make the system asymptotically stable. If $F \notin S_F$, the index 'J' may tend to infinity. Stabilizability and detectability conditions assumed for standard regulator problem do not, in this case, guarantee stability [51]. The existance of a stabilizing feedback 'F' is a non-trivial problem, a satisfactory solution to which is not yet available.

Sufficient conditions for existance of solution to output feedback regulator problem are given in [41], in terms of compactness of the level set as defined below

$$\pi(\mu) = (F \in S_F \mid J(F) \leq \mu) \qquad (7)$$

3.2.2 Discrete Time Systems

Consider the linear discrete time-invariant system described by

$$x_{k+1} = A\, x_k + B\, u_k \tag{8}$$

$$y_k = C\, x_k + D\, u_k \tag{9}$$

$$y_{p_k} = C_p x_k + D_p u_k \tag{10}$$

where

$x_k \in R^n$ is the state vector, $u_k \in R^m$ is control vector, $y_k \in R^1$ is measurement vector and $y_{p_k} \in R^{1_p}$ is regulated output vector. It is assumed that realizations (A, B, C) and (A, B, C_p) are complete and that C and C_p are of full rank.

Input u_k is restricted to be a linear constant feedback of the instantaneously available measurements y_k. Therefore, the set of admissible controls is defined by

$$u_k = -F\, y_k\ , \quad k = 0, 1, \ldots \tag{11}$$

The index of performance of the system is

$$J = \sum_{k=o}^{\infty} \quad y'_{p_k}\ Q_1\ y_{p_k} + 2u'_k\ T_1\ y_{p_k} + u'_k\ R_1\ u_k \tag{12}$$

where

$Q_1 = Q'_1 \geq 0$ and $R = R' > 0$ are symmetric matrices, and $Q_1 - T'_1\, R_1^{-1}\, T_1 > 0$.

Minimization of the performance index (12) is possible if and only if the set of stabilizing feedback gains, S_F shown in (13), is non empty

$$S_F := \left(F \in R^{m \times l} \mid \rho(A-BFC) < 1 \right) \qquad (13)$$

where

$\rho(\cdot)$ denotes spectral radius.

3.3 NECESSARY CONDITIONS FOR SOLUTION TO THE PROBLEM

Performance index (5) is transformed into state regulator problem using (3),

$$J = \int_0^\infty (\underline{x}' Q\underline{x} + \underline{u}' T\underline{x} + \underline{x}' T\underline{u} + \underline{u}' R\underline{u})dt + \gamma(F) \qquad (14)$$

where

$$Q = C_p' Q_1 C_p \quad , \quad T = D_p' Q_1 C_p + T_1 C_p$$

$$R = D_p' Q_1 D_p + T_1 D_p + D_p' T_1' + R_1$$

Using (2) and (4), the state equation (1) becomes

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) - B(I_m + FD)^{-1} F\, C\, \underline{x}(t)$$

Defining $F_1 = (I_m + FD)^{-1} F$ (15)

$$\dot{\underline{x}}(t) = (A - BF_1C)\,\underline{x}(t) \qquad (16)$$

and $u(t) = -F_1C\,\underline{x}(t)$ (17)

Thus solution to the problem for a system having direct feedthrough, exists only if $(I_m + FD)^{-1}$ exists.

Substituting for u in (14) from (17), and using the transition matrix of the system $\phi(t)$, (14) becomes

$$J = \underline{x}_o' \int_0^\infty (\phi'(t)\ (Q+C'F_1'T) + T'F_1C+C'F_1'RF_1C)\ \phi(t))\ dt\ \underline{x}_o + \gamma(F_1) \tag{18}$$

where

$\underline{x}_o = \underline{x}(o)$ is the initial state vector of the system.

For system (16), $\phi(t) = e^{(A-BF_1C)t}$.

Defining

$$S = \int_0^\infty (e^{(A-BF_1C)'t}\ (Q+C'F_1'T+T'F_1C+C'F_1'RF_1C)\ e^{(A-BF_1C)(t)})dt \tag{19}$$

(18) becomes

$$J = \underline{x}_o'\ S\ \underline{x}_o + \gamma(F_1) = Tr(S\ \underline{x}_o\ \underline{x}_o') + \gamma(F_1) \tag{20}$$

where

Tr(.) denotes trace of a square matrix.

S satisfies the Lyapunov equation [29]

$$A_c'S + S\ A_c + W = 0 \tag{21}$$

where

$A_c = A-BFC$ and $W = Q+C'F_1'T+TF_1C+C'F_1'RF_1C$.

Also, from (14), using trace operation

$$J = \int_0^\infty Tr\ ([Q+C'F_1'T+T'F_1C+C'F_1'RF_1C]\ \underline{x}\ \underline{x}')\ dt \tag{22}$$

Interchanging integration and trace operations and substituting $P = \int_0^\infty (\underline{x}\ \underline{x}')\ dt$, (23)

$$J = Tr([Q+C'F_1'T+T'F_1C+C'F_1'RF_1C]\ P) = Tr[WP] \tag{24}$$

where, W is as in

P also satisfies the Lyapunov matrix equation

$$A_c P + P A_c' + \underline{x}_o \underline{x}_o' = 0 \tag{25}$$

where

$$A_c = A - BF_1 C .$$

Proofs for the above and the necessary conditions for the solution to be stated later are given by different authors [29], [33], [46], [63].

So without getting embroiled into the details of proof we can state the necessary conditions, which is given as a theorem.

Theorem 3.3.1 [39]

For the systems (1) - (3), optimal control in the context of optimization of J (5) is given by

$$U_{opt}(t) = -F^* y(t)$$

with

$$F^* = F_1^* (I_1 - DF_1^*)^{-1} \tag{26}$$

where

F_1 satisfies the condition rank $[I - DF_1] = 1$, and

$$F_1^* = R^{-1} ((B'S + T) PC' - 1/2 \gamma'(F_1)) (CPC')^{-1} \tag{27}$$

where

$\gamma'(F_1) = \dfrac{\partial \gamma(F_1)}{\partial F_1}$ and $S = S'$ and $P = P'$ are the solutions of the Lyapunov equations

$$A_c{}'S + SA_c + W = 0 \tag{27a}$$

$$A_cP + PA_c' + \underline{x}_o\,\underline{x}_o' = 0 \tag{27b}$$

with

$A_c = A - BF_1C$ and $W = Q + C'F_1'T + T'F_1C + C'F_1'RF_1C$.

Since $R > 0$, $W > 0$ and S being the solution of Lyapunov equation, for A_c strictly Hurwitz $S > 0$. Similarly, for $\underline{x}_o\,\underline{x}_o' > 0$, $P > 0$.

$\gamma(F_1)$ is used instead of $\gamma(F)$ because of computational ease. For $D = 0$, $F_1 = F$.

Remark 3.3.1

In (3) if $C_p = I$ and $D_p = 0$, the problem becomes a state regulator problem.

$$J = \int_0^\infty (\underline{x}'Q\underline{x} + 2\underline{u}'T\underline{x} + \underline{u}'R\underline{u})\,dt + \gamma(F_1) \tag{28}$$

Remark 3.3.2

In normal cases, the cross-weighting terms are not considered, i.e., $T_1 = 0$.

Then, from (28),

$$J = \int_0^\infty (\underline{x}'Q\underline{x} + \underline{u}'R\underline{u})\,dt + \gamma(F_1) \tag{29}$$

and for $\gamma(F_1) = 0$, (27) become

$$F_1 = R^{-1}B'SPC'(CPC')^{-1} \tag{30}$$

Remark 3.3.3

For systems without direct feedthrough D = O in (2) and $F_1 = F$. Thus the problem reduces to finding the feedback gain matrix F such that

$$\underline{u} = -F\underline{y} = -F\,C\underline{x}\ . \tag{31}$$

Remark 3.3.4

In (27) $(CPC')^{-1}$ exists because $P > 0$ and rank $[C] = l$.

If $l = n$, result is that of optimal regulator with state feedback, with

$$F = R^{-1}\,(B'S + T)\,C^{-1} \tag{32}$$

$$A'S + SA + Q - (B'S + T)'\,R^{-1}(B'S + T) = 0\ .$$

For $T = 0$, and $C = I$

$$F = -R^{-1}\,B'S \tag{33}$$

$$A'S + SA + Q - S'B'R^{-1}BS = 0 \tag{34}$$

which are the well known results of standard LQR theory.

Thus if C^{-1} exists, dependency of F^* on P, cf. (27) and (32), and thus on initial state x_o, cf (27b), disappears.

Remark 3.3.5

Equations (27) and (27b) show that feedback matrix F is dependent on the initial state x_o. Since initial state x_o may not be known a priori, this dependence is undesirable.

As mentioned earlier in Chapters 1 and 2, this problem can be dealt with by considering $\underline{x}_o$ as a random vector with covariance matrix X_{cov}. The problem is then to minimise expected value of cost, J, i.e., to find a control to minimize

$$J = E\left[\int_0^\infty (\underline{y}'_p Q_1 \underline{y}_p + 2\,\underline{u}' T_1 \underline{y}_p + \underline{u}' R_1 \underline{u})\, dt\right] \tag{35}$$

$$= E\,[\underline{x}'_o\, S\, \underline{x}_o] = E\,[Tr\,(S\,\underline{x}_o\,\underline{x}'_o)]$$

$$= Tr(SX_o) \tag{36}$$

where

$X_o = E(\underline{x}_o\,\underline{x}'_o)$ is second order moment of $\underline{x}_o$.

(27b) becomes

$$P\,A_c' + A_c P + X_o = 0 \tag{37}$$

$$X_o = E(\underline{x}_o\,\underline{x}_o') = E((\underline{x}_o-\overline{x}_o)\,(\underline{x}_o-\overline{x}_o)')+\overline{x}_o\,\overline{x}'_o \tag{38}$$

$$= X_{cov} + \overline{x}_o\,\overline{x}'_o$$

where

$$\overline{x}_o = E[\underline{x}_o]$$

For $\overline{x}_o = 0$, $X_o = X_{cov}$

Kuhn et al. [35] suggest X_o to be chosen as $X_o = \text{diag}\,(1/4\,|x_{io}|^2_{max})$ where $|X_{io}|_{max}$ are maximal values for independent identically distributed, initial

states with Gaussian distribution.

Remark 3.3.6

For a stochastic system, with plant disturbances 'ν', and system described by

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} + G\nu \tag{39}$$

$$\underline{y} = C\underline{x}$$

same set of equations as in Theorem 3.3.1 can be used for solution with X_o substituted by GVG', where V is the covariance matrix for the white noise disturbances 'ν'. In this case, the cost (36) become

$$J = Tr\,[S\;GVG'] \tag{40}$$

3.3.2 Discrete time Case

Along the lines of continuous-time case necessary conditions for the solution of the discrete-time case can be obtained as a set of coupled nonlinear matrix equations.

Thus the performance index (12) for T = 0 can be written as

$$J = Tr\,[S\;\underline{x}_o\;\underline{x}_o'] \tag{41}$$

where

$$S = \sum_{k=o}^{\infty} [(A-BF_1C)^k]'\;(Q+C'F_1'RF_1C)\;[(A-BF_1C)^k]$$

is the symmetric positive definite solution of the discrete Lyapunov matrix equation,

$$S = (A-BF_1C)'S\ (A-BF_1C) + Q + C'F_1'RF_1C \tag{42}$$

Using the expression for S in (41) J can also be written as

$$J = Tr\ \lfloor(Q+C'F_1'RF_1C)\ P] \tag{43}$$

where

$$P = \sum_{k=o}^{\infty} \underline{x}_k\ \underline{x}_k' = \sum_{k=o}^{\infty} (A-BF_1C)^k\ \underline{x}_o\ \underline{x}_o'\ [(A-BF_1C)^k]'$$

is the symmetric positive definite solution of the discrete Lyapunov equation.

$$P = (A-BF_1C)\ P(A-BF_1C)' + \underline{x}_o\ \underline{x}_o' \tag{44}$$

The corresponding optimal feedback gain matrix F_1 is given by

$$F_1 = (R+B'SB)^{-1}\ B'SAPC'\ (CPC')^{-1} \tag{45}$$

With cross weighting term T and penalty on F_1, $\gamma(F_1)$,

$$F_1 = (R+B'SB)^{-1}\ [(B'SA+T)\ PC' - 1/2\ \gamma'(F_1)]\ (CPC')^{-1} \tag{46}$$

The equations (45), (44) and (42) constitute the necessary conditions for the existance of optimal output feedback.

All the remarks in the previous section can be considered for discrete-time case too.

The necessary condition presented in Sections 3.3.1 and 3.3.2 are a set of nonlinear coupled equations, solution

of which yields the optimal output feedback. The conditions being only necessary ones and not sufficient, there will be solutions of these equations which are not optimal.

3.4 STRUCTURE CONSTRAINED REGULATORS

Sometimes it becomes necessary to select a particular structure for the feedback gain matrix to make each control variable depend on a different set of output variables. Constrained feedback structure leads to considerable decrease in complexity for minimal deterioration in performance [55]. This is of special significance when some elements of the feedback gain matrix are non-productive [40]. For example, in large systems, non-zero elements of feedback matrix entails all measurements be available to all subsystems, but the cost of the required communication links may outweigh the benefits gained from them.

Several structures can perform close to the unconstrained regulators.

As in the case of unconstrained output feedback, the system may not be stabilizable with some or any of the constrained feedback matrices; only a few of them may be realizable.

If S_{F_c} denotes the set of stabilizing feedback gain matrices with allowable structures,

$$S_{F_c} := (F \in \mathcal{F}_c \mid \rho(A-BFC) \in C^-) \tag{47}$$

where

$\mathcal{F}_c$ is the set of feedback gains with allowable feedback structure.

$$S_{F_c} \subset S_F .$$

3.5 MODIFIED DESCENT ANDERSON MOORE ALGORITHM

The set of equations representing the necessary conditions in Theorem 3.3.1, can be solved in several ways. Levine and Athans [39] who derived these necessary conditions and several others [56], [58] solved the nonlinear set of matrix equations to arrive at a solution. But solving nonlinear equations is difficult and computationally demanding, although the rate of convergence is good as shown in [58], for some examples.

Anderson and Moore [4] solved coupled linear equations iteratively to get the solution. With an initial feedback gain matrix, $F_o \in S_F$ sequence $\{F_k\}$ is generated as,

$$F_{k+1} = -R^{-1}B'S_kP_kC'(CP_kC')^{-1} \tag{48}$$

where

$S_k = S(F_k)$ and $P_k = P(F_k)$ are the solutions of Lyapunov equations (27a) and (27b) respectively.

The above algorithm, however, was found to diverge [56]. An alternate way is to find a sequence of gains $\{F_k\}$ generated by

$$F_{k+1} = F_k + \alpha_k H_k \tag{49}$$

where

$\alpha_k > 0$ is a step length parameter, $F_k \in S_F$ and H_k is a descent direction.

In the algorithm employed here, H_k is selected so as to minimise a quadratic approximation to the loss increment as given in equation (76) [41].

The search direction is a significant factor in deciding the speed of convergence of the algorithm.

α_k decides the step size in each iteration and has to be selected judiciously to improve the efficiency of the algorithm in terms of the number of loss function evaluations, iterations, etc. In this algorithm α_k is selected in each iteration using quadratic fitting, so as to satisfy the Goldstein steplength rule [24].

When the feedback structure is constrained each control input is generated as

$$u_i(t) = -F_i \, \underline{y}_i(t), \qquad i = 1,2, \ldots, m \tag{50}$$

where

$$\underline{y}_i = C_i \, \underline{x}(t) \qquad i = 1,2, \ldots, m$$

C_i is a submatrix of C corresponding to vector $\underline{y}_i$.

Kosut [34] calls this as minimal structure constraints problem.

Each row of the feedback matrix 'F' is given by

$$F_i = \frac{1}{r_i} \; b_i' \; SPC_i' \; (C_i PG_i')^{-1} \tag{51}$$

where

$r_i = R(i,i)$, R is assumed diagonal, and b_i is the ith column of B.

Based on the above discussion a stepwise procedure is given below to iteratively compute a static output feedback gain matrix with and without structure constraints.

Algorithm

Step 1

Choose $C_g \in (0, 1/2)$, $\gamma_o > 0$, $Z_{max} \geq Z_{min} > 1$, $\beta_{min} \leq \beta_{max} < 1$, $J_{max} \geq 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, where ϵ_1 and ϵ_2 are arbitrarily small values to be decided by the designer depending upon the accuracy of solution required.

Set $k = 0$.

Step 2

Find an initial stabilizing feedback F_o with structure constraints.

Step 3

Solve for S and P using (27a) and (27b).

Step 4

Compute

$$(\frac{\partial J}{\partial F})_k = 2[RFCPC' - (B'S+T)\ PC'] + \gamma'(F) \tag{52}$$

$$H_k = \Delta F_k = R^{-1}[(B'S+T)PC' - 1/2\ \gamma'(F)]\ (CPC')^{-1} - F_k \tag{53}$$

For multiple structure constraints,

$$\Delta F_i = \frac{1}{r_i}\ (b_i'S + T)\ PC_i\ (C_iPC_i')^{-1} - F_i \tag{54}$$

where

ΔF_i is the ith row of ΔF.

Step 5

Test for termination.

If $||\Delta F|| < \epsilon_1$, go to step 9 : Otherwise continue.

Step 6

Determine α_k by finite research process.

a1) Set $\alpha = \gamma_k$. Determine smallest integer j,
$0 \leq j \leq j_{max}$, such that $(F_k+\alpha_j\ H_k) \in S_F$ and

$$J(F_k) - J(F_k + \alpha_j\ H_k) \leq -(1-C_g)\ \alpha_j\ Tr\ [(\frac{\partial J}{\partial F})_k'\ H_k] \tag{55}$$

where

$$\alpha_{j+1} = \min\ (\max\ [Z_{min}\ \alpha_j,\ \phi],\ Z_{max}\ \alpha_j) \tag{56}$$

where

$$\phi = \begin{cases} +\infty & \text{if } \Phi \leq 0 \\ -\frac{\alpha_j^2}{2}\ Tr[(\frac{\partial J}{\partial F})_k'\ H_k]\ \Phi^{-1} & \text{if } \Phi > 0 \end{cases} \tag{57}$$

where

$$\bar{\Phi} = J(F_k+\alpha_j H_k) - J(F_k) - \alpha_j \, Tr \, [(\frac{\partial J}{\partial F})_k' H_k] \qquad (58)$$

a2) If $j = 0$ go to step 6b).

If $j \neq 0$ go to 6c.

6b) Set $\alpha = \gamma_k$, determine the smallest j such that $(F_k + \alpha_j H_k) \in S_F$ and

$$J(F_k) - J(F_k+\alpha_j H_k) \geq -C_g \, \alpha_j \, Tr \, [(\frac{\partial J}{\partial F})_k' H_k] \qquad (59)$$

where

$$\alpha_{j+1} = \min (\max [\beta_{min} \alpha_j, \Phi], \beta_{max} \alpha_j) \qquad (60)$$

Φ being given by (57).

6c) Select $\min_{\alpha_j} J(F_k+ \alpha_j H_k)$

Set $\alpha_k = \alpha_{j_{min}}$

Step 7

Test for termination.

If $\dfrac{J(F_k) - J(F_{k+1})}{J(F_k)} \leq \epsilon_2$, go to step 9 ; otherwise continue.

Step 8

Set $F_{k+1} = F_k + \alpha_k H_k$, $k = k + 1$.

Go to Step 3.

Step 9

Set $F^* \longleftarrow F(I-DF)^{-1}$

Remark 3.5.1

$\{\gamma_i\}$ is a series which can be generated with the

relation [42],

$$\gamma_{k+1} = \max\,(\omega\,\gamma_k + (1-\omega)\,\alpha_k,\,\gamma) \tag{61}$$

where

$$0 < \omega < 1,\ \gamma_o = 1,\ 0 < \gamma < 1$$

Or set equal to 1, $\gamma_k \equiv 1,\ \ k = 0,1,\ \ldots$

Remark 3.5.2

In the breakdown of CPU usage, function and gradient evaluation dominate the solution time [23] and these are in turn subject to efficiency of solution of Lyapunov matrix equations.

In the algorithm for solving (27a) and its dual (27b), Bartel and Stewart algorithm [7] is the most efficient method available. The algorithm is a direct procedure to solve equations of the form A'X + XA = -C and is applicable to unstable As too. Bélanger [8] had shown that this algorithm was superior to other existing ones with respect to accuracy and speed. A more efficient algorithm for solving equation of the form A'X + XB = -C is developed by Golub et al. [25]. However, for solving Lypunov equations, there is no advantage over Bartel and Stewart algorithm.

In the algorithm, A is transformed into Schur form, which facilitates calculation of eigenvalues of the system for testing stability, without additional computational effort.

Remark 3.5.3

Using equations (42), (44) and (45) in place of (27a), (27b), and (27) the same algorithm can be extended to the discrete time case.

3.6 INITIAL STABILIZING FEEDBACK

All algorithms require an initial stable feedback gain matrix to initiate the algorithm. The problem of finding such a priming feedback is not, however, trivial.

A stabilizing output feedback gain matrix may be found using pole placement techniques [51] or cost techniques [61] involving minimization of a function of eigenvalues or coefficients of the characteristic polynomial.

The method adopted here is different and is illustrated in a flow diagram in Fig. 1.

3.6.1 Minimum Norm and Minimum Error Excitation

Minimum norm and minimum error excitation are found using the optimal state feedback matrix and the solution of associated algebraic Riccati equation [34]. Since the guarantee that the resulting feedback will be stabilizing is not there, this is only a trial to avoid more computations.

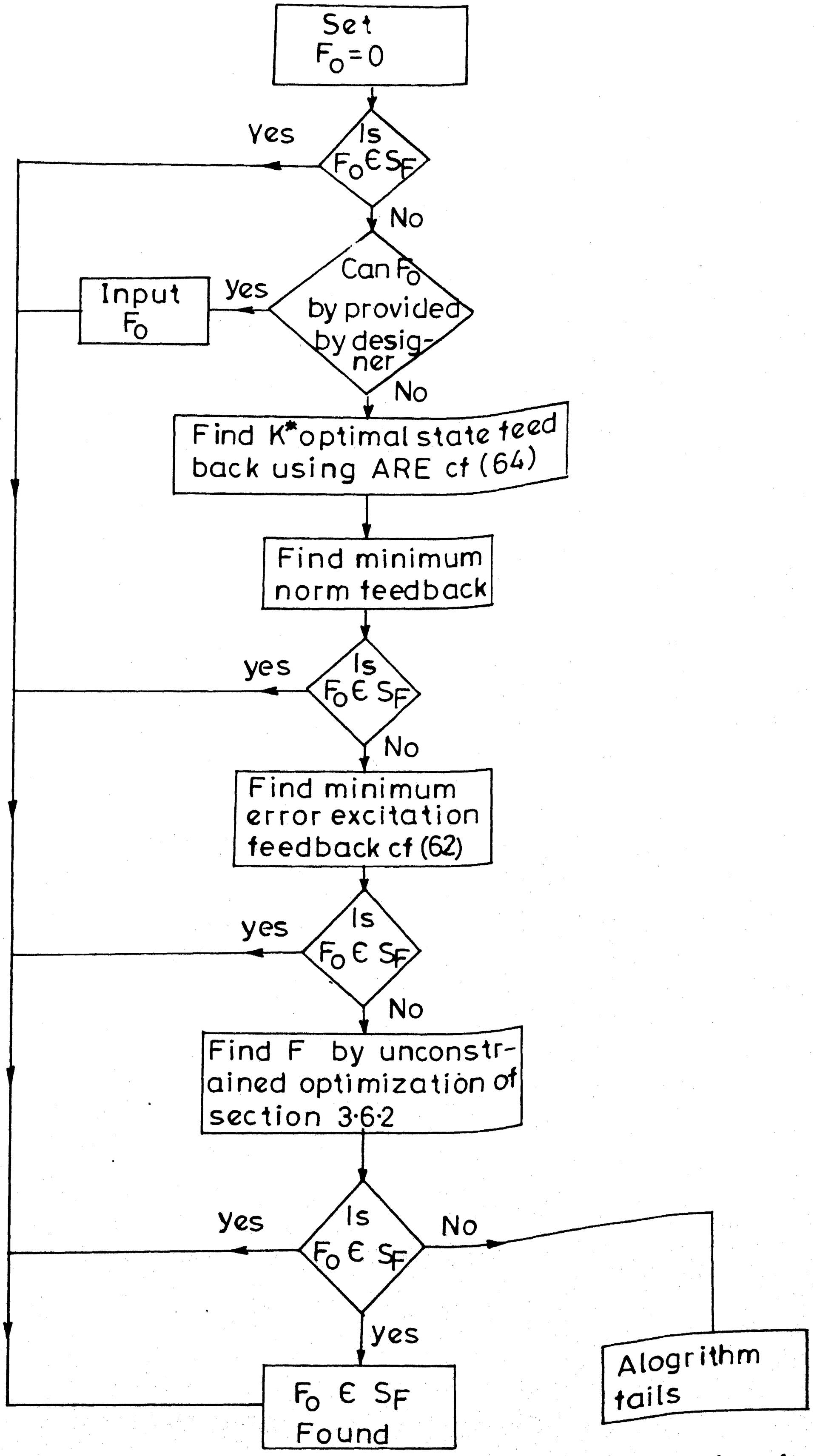

Fig.1 Flow diagram of the method used for finding on initial stabilizing output feedback

For the minimum norm feedback [34] the optimal state feedback solution K* found using (33) and (34) is put in the required output feedback structure form, retaining the unconstrained elements.

Minimum error excitation feedback [34],

$$F_i = K_i^* PC_i' (C_i PC_i')^{-1}, \quad i = 1,2, \ldots, m \tag{62}$$

where C_i is as in (50), K_i^* is ith row of K^* the optimal state feedback and P is the solution of Lyapunov equation,

$$(A-BK^*)P + P(A-BK^*)' + X_o = 0 \tag{63}$$

with $X_o = \underline{x}_o \underline{x}_o'$ if $\underline{x}_o$ is known a priori and $E[\underline{x}_o \underline{x}_o']$ otherwise.

K* is given by,

$$K^* = R^{-1} B'S$$

where

S is the solution of the algebraic Riccati equation,

$$A'S + SA + Q - S'B'R^{-1}BS = 0 \tag{64}$$

Since (62) - (64) contain dynamic information only about the optimal system and not about suboptimal system, stability is not ensured. When above methods fail, constrained optimization discussed in Section 3.6.2 is done to find $F_o \in S_F$.

3.6.2 Constrained Optimization

In this method, an additional constraint is added to the original problem and unconstrained optimization i...

using method of multipliers [26].

In [17], [55] and [62] the same method is used to optimize the constrained cost and find a solution to the problem. In this thesis only partial optimization is done till a stabilizing feedback F_o is obtained.

For required feedback structure F_f, which has free elements and fixed elements, the error $\emptyset(F)$ is defined as

$$\emptyset(i,j) = \begin{cases} F(i,j)-F_f(i,j) & \text{for constrained element in F} \\ 0 & \text{for free elements in F} \end{cases} \tag{65}$$

where

$F_f(i,j)$ correspond to fixed elements in F_f.

Thus cost J cf. (36) is to be minimised subject to equality constraint (65).

This classical equality constrained static optimization problem encountered here is solved with Hestenes method of multipliers [26]. This method is shown to be computationally simple and more stable than penalty function method [30].

Define the augmented Lagrangian,

$$\hat{J} = J + \text{Tr}\,[\Lambda'\,\emptyset] + 1/2\;\gamma\;\text{Tr}\,[\emptyset'\,\emptyset] \tag{66}$$

where Λ is the multiplier matrix, $\gamma > 0$ is a scalar constant and $\emptyset$ is as defined in (65).

For $\Lambda = 0$, J represents a straight penalty method. In this method, γ has to be selected large to force $\emptyset \Longrightarrow 0$. Computational difficulties due to rounding off error and resulting ill-conditioning of associated Hessian matrix arise when $\emptyset$ gets small [62]. Addition of multiplier term alleviates this problem.

The algorithm for solving the unconstrained optimization problem in order to obtain $F_o \in S_F$ is presented below. In the algorithm, Broyden-Fletcher-Goldfarb-Shanno (BFGS method) is used since it is the most efficient method to solve unconstrained optimization problems [19], [35]. It has the advantage of fast convergence rate but disadvantage of requirement of large storage space.

Algorithm

Step 1

Matrix C is augmented by adding rows to make it a non-singular square matrix.

Matrix F is augmented with additional columns to make it confirmable with C.

With any stabilizing full state feedback, K_s

$$F = K_s C^{-1} \quad \text{is a stabilizing matrix} \tag{67}$$

Step 2

Select $\beta_1 > 1$

and set $\gamma \geq 2\beta_1 \, J/Tr\, [\emptyset' \, \emptyset]$ (68)

β_1 determines the emphasis placed early in the descent on satisfying the structural constraints on F. $\beta_1 \geq 1$ so that penalty term should be at least comparable to J.

Select the number of descents in one cycle n_d,

$\beta_2 > 1$ and set $\wedge = 0$, and $i = 1$.

Step 3

Set $H_o = I$

where, I is an identity matrix of order $l_n \times l_n$. $l_n = mxn$ and H is the Hessian matrix.

Step 4

Solve for S and P with (27a), (27b)

Find $\frac{\partial J}{\partial F} = \frac{\partial J}{\partial F} + \wedge + \gamma\, \emptyset(F)$ (69)

where

$\frac{\partial J}{\partial F}$ is given in (52), $\emptyset$ in (65).

Step 5

If $i = 0$, go to Step 6 ;

Otherwise update the Hessian matrix H.

$$H_{i+1} = H_i - \frac{H_i \, q_i p_i' + p_1 q' \, H_i}{p_i' q_i} + [1 + \frac{q_i' H_i q_i}{p_i' q_i}] \, \frac{p_i p_i'}{p_i' q_i} \tag{70}$$

where

$$p_i = col(F_{i+1}) - col(F_i)$$

$$q_i = col(\frac{\partial J}{\partial F})_{i+1} - col(\frac{\partial J}{\partial F})_i$$

where

col(.) is defined as :

$$z = col(\partial J) := z' = [\partial j_1' \quad \partial j_2', \ldots, \partial j_m'] \tag{71}$$

where ∂j_i's are columns of ∂J.

Step 6

Find

$$col(\triangle F_i) = H_i \; col(\frac{\partial J}{\partial F}) \tag{72}$$

where $\triangle F \in R^{mxn}$ is the search direction matrix.

Step 7

Find step size parameter α_k. A method for finding such an α_k is presented below.

Stage 1

a) Set $\alpha_1 = 0$, $J_1 = J(F_1)$, $k = 1$

b) Set $\alpha_k = \alpha_o$, $J_k = J(F_k)$

If $F_k \notin S_F$, reduce α_o to make $F_k \in S_F$.

where, S_F is the set of stabilizing feedback gains.

Set $k = k + 1$.

c) If $J_k \geq J_{k-1}$ go to next step; otherwise reset $\alpha_o = 2\alpha_o$ and go to (b)

d) If $k > 2$ go to next step; otherwise reset

$\alpha_o = \alpha_o/2$ and $k = 1$ and go to (b).

e) Set $A = \alpha_{k-2}$, $B = \alpha_{k-1}$, $C = \alpha_k$ and find

$$\alpha_o = \frac{A+B+C}{3} \text{ or } \frac{B+B+C}{3} \text{ if } \frac{A+B+C}{3} = B.$$

Set $\alpha_s = \alpha_o$, $J_s = J_o$

Stage 2

f) Let $J_B = \min(J_A, J_B, J_C, J_S)$

find 2 adjacent points to B and label them A and C.

$J_A \geq J_B \leq J_C$.

g) Compute $\alpha_o = \frac{A+B+C}{3}$ and $J(\alpha_o)$

Set $\alpha_d = \alpha_o$ and $J_d = J_o$

If $|J_s - J_d| \leq \in J_d$ go to (h) ;

otherwise set $\alpha_s = \alpha_d$, $J_s = J_d$ and go to step (f).

h) Find α_i^* as ,

$$J_{\alpha_i^*} = \min(J_B, J_D)$$

Step 8

Set

$$F_{i+1} = F_i + \alpha_i^* \Delta F_i \quad \text{and } i = i+1$$

Step 9

If $i \neq k\, n_d$ where $k = 1,2, \ldots$ go to step 4;

otherwise continue.

Step 10

If $F \in S_{F_c}$, where S_{F_c} is a set of stabilizing output feedback gains with the required structure. Set $F_o = F$; otherwise update γ and $\wedge$ using (73) and (74) and go to Step 3.

$$\gamma_{j+1} = \beta_2 \gamma_j \tag{73}$$

$$\wedge_{j+1} = \wedge_j + \gamma_j \, \emptyset \, (F_i) \tag{74}$$

Remark 3.6.2.1

Step 10 is included to find a stabilizing F before optimization of cost function J is complete. If desired optimization can be done for more number of cycles, thus partially optimizing J, the original cost, cf. (36).

Remark 3.6.2.2

The search method in Step 7, called arithmetic mean method, is found superior to RMS, quadratic and cubic interpolations and golden section methods in terms of number of function evaluations and CPU time [54].

If this algorithm fails to find a stabilizing feedback the whole method fails.

3.7 SEARCH DIRECTION

The search direction H_k, cf. (31), decides the rate of convergence of the algorithm. If proper search direction is not found, algorithm may not even converge.

When search direction satisfies the descent condition

$$Tr\ (\frac{\partial J}{\partial F})_k' \ \ H_k < 0 \tag{75}$$

the sequence $\left\{ F_k \right\}$ converges in the descent direction H_k [48].

Different methods have different search directions. In all the gradient methods, the search direction is derived from the gradient of the objective function.

The search direction used in the algorithm presented here is also a gradient based one and it is found to minimize a wquadratic approximation of the loss increment shown below. The search direction is also found to satisfy the descent direction condition [48].

The loss increment $J(F+\Delta F)$ can be expressed as [41], [60]

$$J(F+\Delta F) = J(F)+2\ Tr\ [\Delta F'\ [-B'S(F+\Delta F) + RFC]\ PC']$$
$$+\ Tr\ [\Delta F'\ R\Delta FC\ P(F)\ C'] \tag{76}$$

A Taylor series expansion for loss gives,

$$J(F+\Delta F) = J(F) + q(\Delta F) + o(||\Delta F||)^2 \tag{77}$$

where

$$q(\Delta F) = Tr\ [\Delta F'(\frac{\partial J}{\partial F})] + Tr[\Delta F'\ R\Delta FC\ P(F)\ C']$$
$$+\ 2\ Tr\ [\Delta F'B'\Delta S(\Delta F)\ P(F)\ C'] \tag{78}$$

$S(\Delta F)$ is the first order Taylor series form in F of S(F).

A positive definite approximation to $q(\Delta F)$ obtained by ignoring the last term in (78), i.e., taking $S(\Delta F) = 0$ or

$$S(F+\Delta F) = S(F) \qquad (79)$$

gives

$$q(\Delta F) = \mathrm{Tr}\,[\Delta F' (\frac{\partial J}{\partial F})] + \mathrm{Tr}[\Delta F' R \Delta F C\, P(F)\, C'] \qquad (80)$$

The search direction is computed to minimize the increment $q(\Delta F)$ with respect to F.

A necessary condition for above is,

$$\frac{\partial q(\Delta F)}{\partial \Delta F} = 0 \qquad (81)$$

Thus, from (80)

$$\frac{\partial J}{\partial F} + 2\, R \Delta F\, C P(F)\, C' = 0$$

i.e., $$\Delta F = -\, 1/2\; R^{-1}\, (\frac{\partial J}{\partial F})\, (CPC')^{-1} \qquad (82)$$

with $\frac{\partial J}{\partial F} = RFCPC' - B'SPC'$

$$\Delta F = -\, 1/2\; R^{-1}\, B'SPC'\, (CPC')^{-1} - F$$

which is the same given in Theorem 3.3.1.

If $S(F+\Delta F)$ in (76) is approximated by a Taylor series expansion, another set of equations involving $S(\Delta F)$ and $P(\Delta F)$ are obtained. Newton's method of solving parameter linear quadratic control problems [60] uses these sets of equations for descent direction in each iteration. Although

Newton's method has quadratic convergence rate, it may be expensive and benefits being only local, may not be effective when far away from solution.

Descent Anderson Moore method with finite search process finds a local minimum to J in (36) if one exists, as stated in the theorem given below.

Theorem 3.7.1 [41]

For nonempty S_F in (6) and compact level set (7), sequence $\{F_k\}$ generated by search direction ΔF in (82), and step length parameter α_k in (49) which is chosen so as to satisfy Goldstein step-length rule with $F_{k+1} \in S_F$, will converge to F_k^* such that

$$\left(\frac{\partial J}{\partial F_k^*}\right) \approx 0 \quad \text{for some } k.$$

Since S_F is an open set, any local minimizer of J(F) on S_F must be a stationary point of loss function J(F) on S_F.

Descent Anderson Moore method has been widely used [15], [42], [48], [59] and [60]. It is shown in [59] that this method is superior in terms of number of iterations and number of loss function evaluations to some of the general purpose algorithms like steepest descent method, BFGS method with and without scaling of initial Hessian matrix H_o.

Since the problem may have several local minima, for example, in the case when S_F is a set of disconnected regions, algorithm will have to be initiated from different initial values of feedback matrix to find the global minimum.

3.7.1 Unidimensional Line Search

Like research direction matrix, step size scaling is also an important factor in deciding the convergence and efficiency of the algorithm. An example which was found to diverge in [56] where step size scaling is not done, can be shown to converge with step size scaling.

The algorithm presented in this thesis uses quadratic fitting with Goldstein step length rule and a modification of Armijo line search [6], for unidimensional search, cf. Step 6 in Section 3.5.1.

Quadratic interpolation is done with three values known at the beginning of the search,

(1) $J(F_k)$ (2) $\mathrm{Tr}\,[(\frac{\partial J}{\partial F})_k' \,\Delta F_k]$ and (3) $J(F_k + \Delta F_k)$

(3) above is the cost at Newton step, $\alpha_k = 1$.

New value α is given by [19], [42]

$$\alpha = \frac{-\,\alpha^2\ \mathrm{Tr}\,[[\partial J/\partial F)_k' \Delta F_k]}{2[J(F_k+\Delta F_k) - J(F_k) - \mathrm{Tr}\,[(\partial J/\partial F)_k'\ \Delta F_k]]} \qquad (83)$$

The step size 'α' chosen should satisfy the Goldstein rule [24],

$$-C_g\ \alpha_k\ \mathrm{Tr}[(\partial J/\partial F)_k' \Delta F_k) < J(F_k) - J(F_k+\alpha_k \Delta F_k)$$

$$\leq (1-C_g)\ \alpha_k\ \mathrm{Tr}[(\partial J/\partial F)_k' \Delta F_k] \tag{84}$$

The left hand side of the above ensures that average rate of descent is at least some prescribed fraction of the initial rate of descent in the search direction, thus avoiding very small decrease in cost values relative to the length of steps taken.

Figure 2 shows the significance of judicious choice of C_g for greater descent. The best value of C_g, however, depends upon the particular problem.

The sequence $\{F_k\}$ generated by using step sizes that satisfy the left hand side of (84), under certain assumptions, will converge to F^*, a local minimiser of cost J. These conditions were demonstrated separately by Armijo [6] and Goldstein [24].

Unlike exact line searches, viz., Golden section method, Newton's method, curve fitting methods like cubic and quadratic interpolations, this method is an inexact line search without any minimization of function of single variable. Although the exact line search gives a parameter for which cost is less than that for the parameter obtained by inexact line searches, the total number of function evaluations may be more.

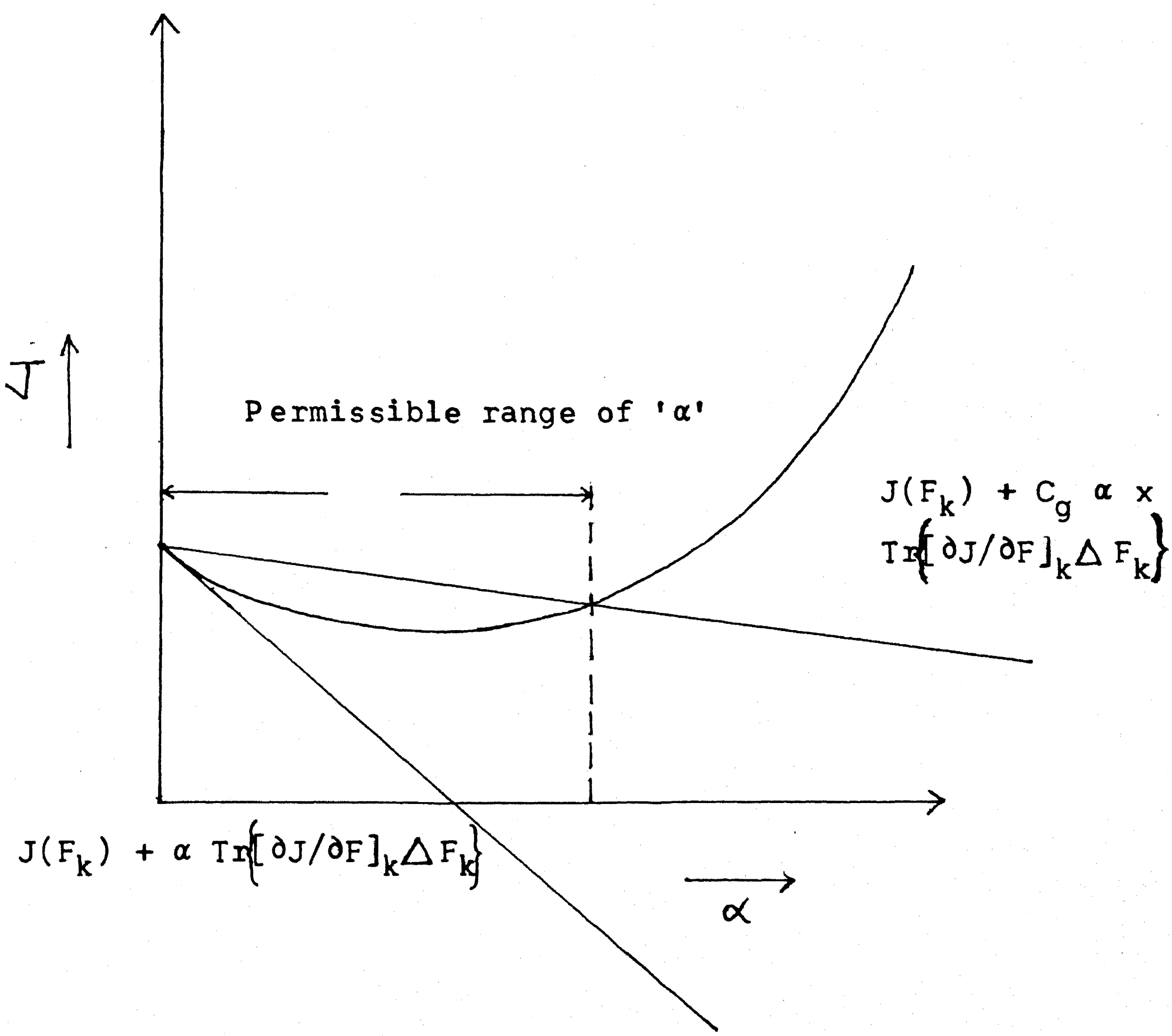

Fig. 2 Illustration of dependence of cost on parameter C_g

3.8 CONVERGENCE CRITERION

One of the following convergence criteria is employed in almost all algorithms.

1) $||\Delta F|| \leq \epsilon$ (85)

2) $||\partial J/\partial F|| \leq \epsilon$ (86)

3) $\dfrac{J(F_{k+1}) - J(F_k)}{J(F_k)} < \epsilon$ (87)

where $||X|| := [\mathrm{Tr}\ (X'X)]^{1/2}$, $\epsilon > 0$ is a small value.

Algorithms in [15], [58], use (85) and those in [16], [35], [42] and [59] use (86).

For special cases, when $J(F_k) \dashrightarrow 0$, (87) should be

$$|J(F_{k+1}) - J(F_k)| < \epsilon \quad (88)$$

Such a criterion is used in [62].

It is suggested to use several criteria simultaneously for a general purpose algorithm [19], since different criteria will be suitable for different problems. Value of ϵ is to be selected according to the accuracy requirement and effort demanded of the program in satisfying it. In the algorithm used here, (85) and (87) are used for testing convergence, cf. Section 3.5.

3.9 CONCLUSIONS

Several algorithms for the design of LQR with output feedback have been studied in this chapter.

CHAPTER 4

DYNAMIC OUTPUT FEEDBACK

4.1 INTRODUCTION

Traditional methods of servomechanism employed dynamic compensators like lag, lead or lag-lead networks to improve system performance. They were designed using root-locus and/or frequency domain methods. These graphical methods were suitable for single-input single-output systems but difficult to handle for multivariable systems.

In the state-space approach, Kalman filter or Luenberger observer is used to estimate the states for feedback, depending upon whether the system is stochastic or deterministic. But for high-order systems, order of these compensators are also high bringing complexity into the system. For the design of Kalman filter, covariance matrices for measurement and plant driving noises have to be specified which, however, is difficult for essentially deterministic systems.

Compensators of order (n-l), where n is the number of states and l the number of outputs, are required for state estimation in reduced-order observers. For designing optimal feedback systems, a minimal order observer which estimates

a specified linear functional of system states can be used. Order of minimal order observers is less than that of reduced order observers. Observers require input to the system for its operation whereas there is no need for it for dynamic output feedback compensators. The dynamic output regulator approach avoids putting poles of compensators at infinity which is allowed in observers, thus not permitting high frequency plant driving noise to pass through.

4.2 DYNAMIC OUTPUT FEEDBACK REGULATOR PROBLEM

Consider a linear time-invariant continuous-time system described by the state and output equations,

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1}$$

$$\underline{y} = C\underline{x} + D\underline{u} \tag{2}$$

$$\underline{y}_p = C_p\underline{x} + D_p\underline{u} \tag{3}$$

where $\underline{x} \in R^n$ is the state vector, $\underline{u} \in R^m$ the input vector, $\underline{y} \in R^l$ the measured output vector and $\underline{y}_p \in R^{l_p}$ the regulated output vector of the system.

Matrices A, B, C, D, C_p and D_p are fixed and of compatible dimensions.

The system is to be regulated by a dynamic compensator of the form

$$\dot{\underline{x}}_c = A_c\,\underline{x}_c + B_c\,\underline{y} \tag{4}$$

$$\underline{u} = C_c\,\underline{x}_c + D_c\,\underline{y} \tag{5}$$

where $\underline{x}_c \in R^{n_c}$ is the state vector of the compensator and A_c, B_c, C_c and D_c are constant matrices of compatible dimension to be determined such that the index of performance

$$J = E\,[\int_0^\infty (\underline{y}_p'\ Q\ \underline{y}_p + \underline{u}'\ T\ \underline{y}_p + \underline{u}'\ R\ \underline{u} + \underline{x}_c'\ Q_{c1}\ \underline{x}_c$$

$$+ \underline{x}_c'\ Q_{c2}\ \underline{y}_p + \underline{y}_p'\ Q_{c3}\ \underline{x}_c + \dot{\underline{x}}_c'\ R_c\ \dot{\underline{x}}_c)\ dt] \qquad (6)$$

where $Q = Q' \geq 0$, $R = R' > 0$, $Q_{c1} = Q_{c1}' \geq 0$ and $R_c = R_c' > 0$ is minimised.

A dynamic output feedback controller is illustrated in Figure 1. The order of the compensator n_c can be chosen with the obvious restriction that a controller of the chosen order must be able to stabilize the plant.

For solving the dynamic regulator problem described above, it is first transformed into the static constant output feedback problem as described below.

4.3 EQUIVALENT STATIC OUTPUT FEEDBACK PROBLEM

The system state vector $\underline{x}$ is augmented with the compensator state vector $\underline{x}_c$. Inputs to the augmented systems are $\underline{u}$ and $\dot{\underline{x}}_c$ and outputs are $\underline{y}$ and $\underline{x}_c$.

Thus defining $\hat{\underline{x}} = \begin{bmatrix} \underline{x} \\ \underline{x}_c \end{bmatrix}$, $\hat{\underline{u}} = \begin{bmatrix} \underline{u} \\ \dot{\underline{x}}_c \end{bmatrix}$, $\hat{\underline{y}} = \begin{bmatrix} \underline{y} \\ \underline{x}_c \end{bmatrix}$, $\hat{\underline{y}}_p = \begin{bmatrix} \underline{y}_p \\ \underline{x}_c \end{bmatrix}$

the system equations (1) - (3), and compensator equations (4) and (5) can be combined as

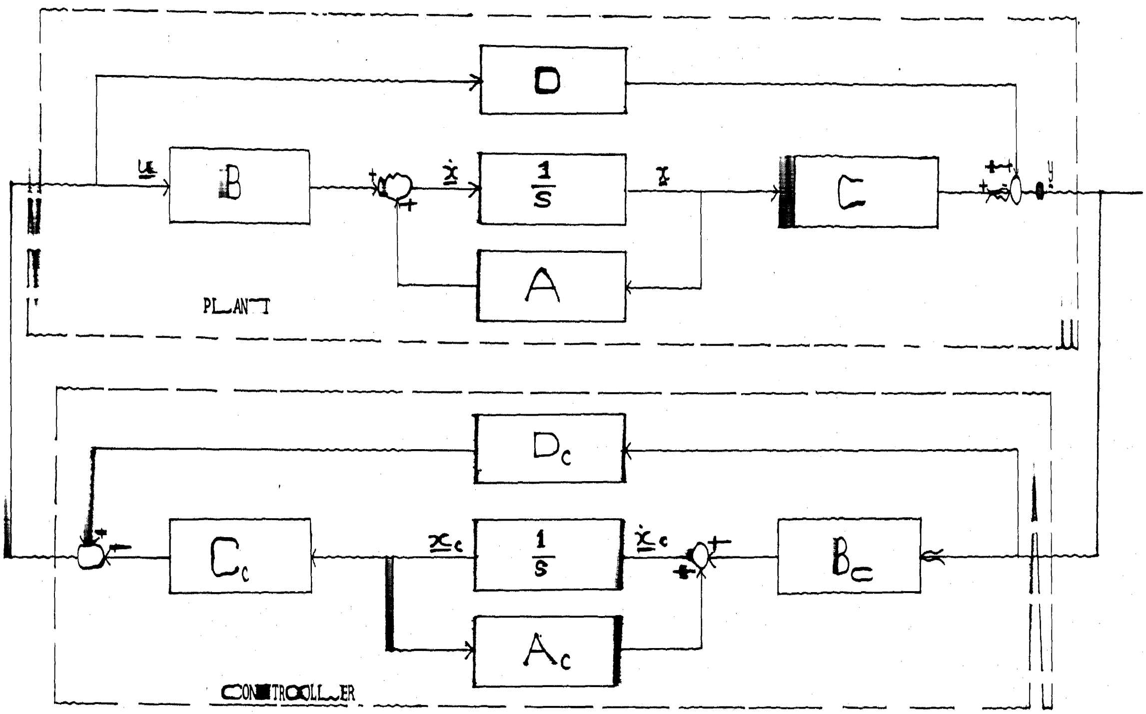

Fig. 1 Block diagram of a system with dynamic output feedback compensator

$$\dot{\hat{\underline{x}}} = \hat{A}\ \hat{\underline{x}} + \hat{B}\ \hat{\underline{u}} \tag{7}$$

$$\hat{\underline{y}} = \hat{C}\ \hat{\underline{x}} + \hat{D}\ \hat{\underline{u}} \tag{8}$$

$$\hat{\underline{y}}_p = \hat{C}_p\ \hat{\underline{x}} + \hat{D}_p\ \hat{\underline{u}} \tag{9}$$

where

$$\hat{A} = \begin{bmatrix} A & O \\ O & O \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B & O \\ O & I \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} C & O \\ O & I \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$$

$$\hat{C}_p = \begin{bmatrix} C_p & O \\ O & I \end{bmatrix}, \quad \hat{D}_p = \begin{bmatrix} D_p & O \\ O & O \end{bmatrix}$$

The control to be realised become,

$$\hat{\underline{u}} = -F\ \hat{\underline{y}} \tag{10}$$

where

$$F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$$

and the index of performance (6) reduces to

$$J = E\left[\int_0^\infty (\hat{\underline{y}}_p'\ \hat{Q}\ \hat{\underline{y}}_p + \hat{\underline{u}}'\ \hat{T}\ \hat{\underline{y}}_p + \hat{\underline{u}}'\ \hat{R}\ \hat{\underline{u}})\ dt\right] \tag{11}$$

where $\hat{Q} = \begin{bmatrix} Q & Q_{c3} \\ Q_{c2} & Q_{c1} \end{bmatrix} \geq 0$, $\hat{R} = \begin{bmatrix} R & 0 \\ 0 & R_c \end{bmatrix} > 0$

and $\hat{T} = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$

When $\underline{x}_o$ and $\underline{x}_{co}$, initial states of the system and compensator are assumed to be random vectors with

$$X_o = E\,[\underline{x}_o\ \underline{x}_o'], \quad X_{co} = [\underline{x}_{co}\ \underline{x}_{co}'] .$$

Second order statistics of initial states of augmented system,

$$X_o = \begin{bmatrix} X_o & 0 \\ 0 & X_{co} \end{bmatrix}$$

, where $\underline{x}_o$ and $\underline{x}_{co}$ are assumed independent.

It can be seen from system equations (7) - (9), control equation (10) and performance index (11) that the problem now is equivalent to a static output feedback problem with feedback gain matrix

$$F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$$

.

This problem can be solved using techniques described in Chapter 3.

While designing dynamic regulators in practical situations with realistic plants, certain difficulties arise [35].

One such difficulty is due to the redundancy in the number of parameters free to be designed. State-space description, with elements of A_c, B_c, C_c and D_c, contains more free elements than the minimum number required to present an arbitrary input-output behaviour. When number of free parameters are more, computational effort required is more. Constraining certain elements in the compensator matrices alleviates this problem to some extent.

In certain cases an optimal regulator is resulted which is partly decoupled from the plant. If the algorithm is started with a lower order compensator, the optimal solution cannot be a higher order compensator, since a minimum of J exists for every compensator order n_c [35]. With the particular structure of block matrices in (7) - (10) time required for computation is more. Use of penalty on feedback elements will particularly be useful in the design of dynamic compensators.

4.4 CONCLUSIONS

The problem of designing LQ regulator using dynamic output feedback is presented in this chapter. Computational difficulties that arise during the design of dynamic output feedback regulators are also discussed.

CHAPTER 5

GUIDELINES FOR USE OF THE DESIGN PACKAGE

Programs using the algorithms presented in the previous chapters have been coded in Fortran code. It can be used for designing linear quadratic regulators with state feedback or output feedback, with or without contraints on the feedback structure. The program is user-oriented, with fixed inputs to be provided for the problem specification and design parameter to be chosen by the user, Since the most suitable values of these parameters differ for different problems, the designer may have to try more than once to get a fast convergent solution. Since the solution may correspond to a local minimum, the algorithm should be initialized from different values of the initial stabilizing feedback gain matrix F_o, to get a global minimum.

Following are the inputs required to specify the problem. These are provided at the start of the program.

n_p = order of the plant

m_p = number of inputs to the plant

l_c = number of outputs of the plant used for controlling the plant

l_p = number of outputs to be regulated

N_{sig} = number of significant digits in the elements of A which determines the convergence criterion ϵ_3 for the reduction of closed loop matrix A_c to Schur form. $\epsilon_3 \leq 10^{-N_{sig}}$

A = system matrix of order $n_p x n_p$

B = input matrix of order $n_p x m_p$

C = measurement output matrix of order $l_c x n_p$

D = direct feed through to measured outputs of order $l_c x m_p$

C_p = regulated output matrix of order $l_p x n_p$

D_p = direct feed through to regulated outputs of order $l_p x m_p$

Q = weighting matrix on regulated outputs of order $l_p x l_p$

T = cross weighting matrix on regulated outputs and controls, of order $m_p x l_p$

R = weighting matrix on controls of order $m_p x m_p$

X_o = $E(x_o\ x_o')$ - second order statistics of the initial state vector

The user can decide the order of compensator n_c. When n_c is given as zero, static optimal output feedback gains are found by the program. When $n_c > 0$, following matrices should be given by the user.

Q_{c3} = cross weighting of system states and compensator states; of order $n_p x n_c$

Q_{c2} = cross weighting of compensator states and system states, of order $n_c x n_p$

Q_{c1} = weighting matrix on the compensator states, of order $n_c x n_c$

R_c = weighting on $\dot{x}_c$, where $\underline{x}_c$ are the compensator states

X_{co} = second order statistics of the compensator states.

The user can decide the penalty on feedback matrix $\gamma(F)$. The penalty function employed in the program is of the form

$$\gamma(F) = \sum_{i=1}^{m} \sum_{j=1}^{l} L(i,j)\,[F(i,j)]^2$$

where F is the feedback gain matrix and L is a matrix of order mxl to be provided by the user. $m = m_p+n_c$ and $l = l_c+n_c$ are the number of inputs and number of outputs of the augmented system. Each element of L decides the penalty on the corresponding element of F.

The design parameters to be provided are given below with recommended values and recommended range of values for each [42].

C_g = This value determines the step length in each iteration of the algorithm.

Although in Goldstein's criterion, $C_g \in (0, 1/2)$, recommended values are $C_g \in (0.05, 0.3)$.

Z_{max} = 4(2,5) and Z_{min} = 2(1.5, 3) such that $Z_{max} \geq Z_{min} > 1$.

These values are used for quadratic fitting in Step 6a1 of algorithm in Sec. 3.5.

J_{max} = 2(0,3) decides the number of times (55) in Chapter 3 is checked. When $J_{max} = 0$, the step 6a1 is skipped which, however, does not affect the convergence property of the algorithm.

β_{min} = 0.2(0.1, 0.3) and β_{max} = 0.35(0.2, 0.5) such that $\beta_{min} \leq \beta_{max} < 1$.

These values are used for quadratic fitting in Step 6b of algorithm in Section 3.5.

ω and γ = such that $0 < \omega, \gamma < 1$. These values are used to update the sequence γ_k.

ϵ_1 = A small parameter which decides the convergence criterion using $||\Delta F_k||$, cf. (53) in Section 3.5.

Value of ϵ_1 that should be given depends upon the accuracy wanted by the user, $\epsilon_1 > 0$.

ϵ_2 = A small parameter which decides the convergence criterion using normalised change in cost for two consecutive iterations, $\epsilon_2 > 0$. Since suitable values of this parameters vary widely for different problems, care should be taken in deciding the value.

CHAPTER 6

RESULTS AND DISCUSSION

It is shown, in this chapter, how the method employed in this thesis fares with other approaches to the problem and other existing methods of solving the problem. The importance of choosing appropriate values for certain design parameters is also discussed. The system description, the weighting matrices etc. for the examples discussed in this chapter are given in the Appendix.

It was observed in Chapter 2 that Therapos [57] used approximation of state feedback closed-loop transfer function to find an output feedback regulator. For the system described in Example 1, he obtained a feedback matrix F_T shown in Table 1. S_T is the corresponding cost matrix. Using the method in this thesis, an optimum feedback F^* was obtained with the corresponding cost matrix S^*. It can be observed that but for a few elements, S^* has elements with values smaller than that of S_T. Thus for almost all initial-state conditions, costs will be lesser when F^* is used for feedback.

But as shown by Allwright [1] the minimization of expected value of cost does not guarantee a closed-loop system which can perform better than the open-loop system

for all initial state conditions. The cost matrix for the open-loop system, S^o, and that for the closed-loop system with optimal output feedback, S*, are shown in Table 2 for Example 2. For an initial state [1 0 0], open-loop cost is less than that of the optimal closed-loop system. The closed-loop system obtained by Allwright [3] after minimizing a performance index

$$J = [\mathrm{Tr}\ ([S(F)]^p]^{1/p} \quad \text{for} \quad p = 11,$$

suffers from the same drawback, as shown in Table 2 for $x_o = [0 \ \ 1 \ \ 0]$. A method to find a dominant feedback, a feedback which results in a closed-loop system with an index of performance better, for all permissible initial state conditions, than that for a closed-loop system formed by any other stabilizing feedback, does not exist, at present.

The descent Anderson, Moore (DAM) method with a good line search has shown good convergence properties. Table 3 and Table 4 shows this algorithm being compared to the Levine-Athans method and its dual algorithm in [58] for Examples 3 and 4. DAM method shows better convergence than both algorithms for Example 3 and than Levine-Athans algorithm for Example 4. Dual Levine-Athans method converges fast in the first few iterations after which it is the same as that for DAM method. Davidon-Fletcher-Powell (DFP) algorithm

employed by Choi and Sirisena [16] to find the optimal output feedback solution for Example 4, took 35 iterations to reach the convergence level $\sum_i (\partial J/\partial F_i)^2 < 0.1$, with an index of performance 159.12. With the same initial feedback gain matrix and convergence level, DAM took only 10 iterations to converge to an optimal cost of 159.09.

Table 5 compares DAM method with steepest descent method which uses gradient of cost with respect to the feedback gain matrix as the search direction, for Example 4.

Using the modification given by Kosut [34], for diagonal R (the weighting matrix on inputs), the algorithm is adapted to finding optimal structure constrained feedback. For Example 5, used by Moerder et al. [48] to illustrate use of penalty on feedback elements, it took only 12 iterations to converge to the feedback gain matrix given in [48]. The penalty method of forcing feedback elements to zero is similar to straight penalty method and may involve computational difficulties.

For unstable systems as well as stable open loop systems, minimum norm and minimum error excitation feedbacks can be good priming feedback gain matrices. For Example 1, when it took 7 iterations with initial feedback gain matrix $F_o = 0$, it took only 3 iterations when minimum norm feedback was used as F_o. For example 7 with F_o same as the one used in [59] (given in Table 9), it took 5 iterations and

20 function evaluations whereas with minimum norm feedback as F_o, cf. Table 9, it took 4 iterations and 15 cost function evaluations. For the unstable system described by Example 8, minimum norm feedback was found to be effective as F_o, the algorithm converging in 9 iterations. Same problem was solved by Choi and Sirisena [17] using unconstrained optimization. Since details regarding number of iterations, number of function evaluations etc. are not given in [17], comparison was not possible.

The algorithm presented in this thesis requires an initial stabilizing feedback having the required structure, which can be obtained using methods presented in Section 3.6. Using minimum norm and minimum error excitation feedbacks, cf. Section 3.6.1, first level unconstrained optimization done in [62] can be avoided in some examples.

DAM algorithm, like other algorithms, shows poor convergence properties for dynamic output feedback regulator problems. It was observed that with $B_{co} = C_{co} = 0$, where B_{co} and C_{co} are the initial compensator matrices B_c and C_c in F_o, the solution also had $B_c^* = C_c^* = 0$, i.e., the compensator was decoupled from the plant. The dynamic compensator obtained for Example 5 are shown in Table 10. Results show that when $B_{co} = C_{co} \neq 0$, $B_c^* = C_c^* \neq 0$. Since weighting is done on compensator states and its derivatives, optimum value of cost may be larger with higher order compensators

but the associated closed loop system may have better response characteristics.

Several parameters are to be provided by the designer in the modified DAM algorithm, cf. Chapter 5. One of them is C_g which determines the Goldstein's step length criterion. This value affects the total number of cost function evaluations to a great extent. Tables 6 and 7 show number of iterations, number of times matrix Lyapunov equation (MLE) is solved and CPU time for different values of C_g, for examples 3 and 5, respectively. Table 7 shows, in addition, cost $J(F_k)$ and $||\Delta F_k||$ at $k = 5$ and $k = 10$. Optimum value of C_g for Example 3 is near 0.2 whereas for Example 5, a value near 0.03 gives better performance. It can also be noted from Tables 6 and 7 that CPU time is directly proportional to the number of matrix Lyapunov equations (MLE) solved.

$\{\gamma_k\}$ is a series which can be generated using the relation (61) in Chapter 3 or set equal to 1 for all iterations. Table 8 compares the two options when $C_g = 0.2$ for Examples 3, 4 and 5. Example 4 favours keeping $\gamma_k \equiv 1$ whereas other two take less number of iterations with γ_k generated by (61). If in Step 6a of the algorithm in Section 3.5, $j > 0$ for a good number of iterations, it is preferred to keep $\gamma_k \equiv 1$.

When second order statistics of $\underline{x}_o$, which is modelled as random vector, are unknown, X_o becomes a design criterion.

TABLE 1 Feedback and cost matrices with Example 1 for comparison with transfer function approach

$$F_T = [0.9245 \quad 0.1705 \quad 0.0406]$$
$$F^* = [0.6229 \quad 0.1090 \quad 0.0241]$$

$$S_T = \begin{bmatrix} 0.3934 & 0.03945 & 0.00389 & 0.01067 & 0.03060 \\ 0.03945 & 0.00574 & 0.00062 & 0.00185 & 0.00518 \\ 0.00389 & 0.00062 & 0.00016 & 0.00023 & 0.00025 \\ 0.01067 & 0.00185 & 0.00023 & 0.00080 & 0.00234 \\ 0.03060 & 0.00518 & 0.00025 & 0.00234 & 0.01760 \end{bmatrix}$$

$$S^* = \begin{bmatrix} 0.35750 & 0.03457 & 0.00270 & 0.00787 & 0.03321 \\ 0.03457 & 0.00432 & 0.00039 & 0.00125 & 0.00634 \\ 0.00270 & 0.00039 & 0.00004 & 0.00011 & 0.00037 \\ 0.00787 & 0.00125 & 0.00011 & 0.00040 & 0.00179 \\ 0.03321 & 0.00634 & 0.00037 & 0.00179 & 0.01347 \end{bmatrix}$$

$Tr\ [S_T] = 0.41765$

$Tr\ [S^*] = 0.38636$

Optimal cost with state feedback, for $E\ [x_o\ x_o'] = I$,

$= 0.37768$

TABLE 2 Illustration of a non-dominant optimal solution using Example 2

$$F_o = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \qquad S^o = \begin{bmatrix} 7.143 & 0.000 & 0.000 \\ 0.000 & 5.000 & 0.000 \\ 0.000 & 0.000 & 25.000 \end{bmatrix}$$

$$F^* = \begin{bmatrix} 0.0794 & 0.3486 \\ -0.1133 & 1.2660 \end{bmatrix} \qquad S^* = \begin{bmatrix} 8.187 & -1.417 & 0.163 \\ -1.417 & 2.825 & -0.616 \\ 0.163 & -0.616 & 4.172 \end{bmatrix}$$

$$F_A = \begin{bmatrix} 0.017 & 0.180 \\ 0.044 & 1.340 \end{bmatrix} \qquad S_A = \begin{bmatrix} 7.087 & -0.270 & -0.050 \\ -0.270 & 5.557 & -0.660 \\ -0.050 & -0.660 & 4.105 \end{bmatrix}$$

For

$$x_o = [1 \quad 0 \quad 0] \,, \qquad J^o = 7.143 \,, \qquad J^* = 8.187$$

For

$$x_o = [0 \quad 1 \quad 0] \,, \qquad J^o = 5.000, \qquad J_A = 5.557$$

TABLE 3 Comparison with modified Levine-Athans Algorithms for Example 3

	Dual Levine-Athans Algm.		Levine-Athans Algm.		Modified DAM Algm.	
k	$J(F_k)$	$\|\|\Delta F_k\|\|$	$J(F_k)$	$\|\|\Delta F_k\|\|$	$J(F_k)$	$\|\|\Delta F_k\|\|$
0	2894.1	170	2894.1	97	2894.1	98.57
1	1866.2	2.2	2194.4	42	1885.9	37.97
2	1764.4	3.7	1896.4	15	1754.6	11.51
3	1745.3	2.4	1787.4	4.9	1740.1	0.74
4	1740.3	1.3	1752.5	1.6	1738.05	0.33
5	1738.7	0.67	1742.3	0.55	1737.94	0.22
6	1738.3	0.36	1739.8	0.21	1737.93	0.05
7	1738.2	0.19	1738.5	0.08	1737.92	0.04
.						
.						
.						
6	1738.2	$<10^{-3}$	1738.2	$< 10^{-3}$	1737.92	$< 10^{-3}$

nitial feedback gain matrix $F_o = [-25 \quad -285]$

ptimal output feedback gain matrix $F^* = [-1.366 \quad -12.42]$

TABLE 4 Comparison with modified Levine-Athans methods for Example 4

	Dual Levine-Athans Algm.		Levine-Athans Algm.		Modified DAM Algm.		
k	$J(F_k)$	$\|\|\Delta F_k\|\|$	$J(F_k)$	$\|\|\Delta F_k\|\|$	$J(F_k)$	$\|\|\Delta F_k\|\|$	$\|\|\partial J/\partial F\|\|$
0	31135	10.0	31135	3600	31135	3537.4	8.5 E+07
1	176.97	1.1	15590	1800	598.1	21.3	2186.82
2	162.22	0.42	7838.6	900	240.6	4.85	110.28
3	159.42	0.23	3981.3	450	189.0	3.27	41.56
4	159.34	0.13	2065.8	220	164.8	2.1	22.24
5	159.16	0.075	1109.9	100	160.2	0.64	6.72
6	159.10	0.044	630.4	56	159.35	0.17	2.31
7	159.08	0.028	389.0	27	159.19	0.08	1.33
.							
.							
.							
?	159.07	$<10^{-3}$	159.07	0.012	159.068	$<10^{-3}$	0.01

itial feedback gain matrix $F_o = 0$

timal output feedback gain matrix $F^* = \begin{bmatrix} -0.398 & -1.592 & -7.852 \\ 1.257 & 3.480 & 5.004 \end{bmatrix}$

timal cost with state feedback = 151.246

TABLE 5 Comparison with steepest descent method for Example 4

	Steepest Descent Method	Modified DAM Method
k	$J(F_k)$	$J(F_k)$
0	31135.1	31135.1
1	8143.8	589.1
2	2395.5	240.6
3	1201.4	189.0
4	900.9	164.8
5	738.8	160.2
6	629.8	159.3

TABLE 6 Results for various C_g in Example 4

C_g	No. of iterations	No. of times MLE is solved	CPU Time (in sec.)
0.5	9	44	1.27
0.4	9	43	1.25
0.3	9	43	1.25
0.2	11	34	1.17
0.1	11	34	1.18
0.04	17	76	2.02

TABLE 7 Results for various C_g in Example 5

C_g	$J(F_5)$	$\|\|\Delta F_5\|\|$	$J(F_{10})$	$\|\|\Delta F_{10}\|\|$	No.of iter.	No.of times MLE is solved	CPU time (in sec.)
0.5	135.54	3.68	133.7	0.082	13	72	9.56
0.3	135.54	3.68	133.7	0.082	13	67	9.05
0.1	135.54	3.68	133.7	0.040	13	60	8.32
0.04	134.27	1.60	133.7	0.02	12	55	7.68
0.02	134.27	1.60	133.7	0.02	12	55	7.68
0.01	196.27	12.6	134.2	2.16	16	69	9.55

TABLE 8 Comparison of $\gamma_k \equiv 1$ and $\gamma_k \not\equiv 1$ using Examples 3,4 and 5

Example	$\gamma_k \equiv 1$		γ_k generated using (61) in Chap. 3	
	No.of iter.	No. of LME calculations	No. of iter.	No. of LME calculations
3	12	76	11	34
4	9	39	11	46
5	16	79	13	62

TABLE 9 Illustration of advantage of minimum norm feedback using Example 7

$$F_o = \begin{bmatrix} 1.151 & -0.56 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.3 & 0.62 & 0 & 0 \\ 0 & 0 & 0.82 & 3.41 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.96 & 0.06 \end{bmatrix}$$

Number of cost function evaluations = 20

Minimum norm feedback as F_o =

$$\begin{bmatrix} 1.222 & 0.469 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.159 & 0.5165 & 0 & 0 \\ 0 & 0 & 0.588 & 2.248 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.168 & 0.322 \end{bmatrix}$$

Number of cost function evaluations = 15

TABLE 10 Dynamic compensators for Example 5

$Q_{c2} = Q_{c3} = 0, \quad Q_{c1} = R_c = X_{co} = I$

For $n_c = 0$,

$$F^* = \begin{bmatrix} -2.601 & -0.396 & 2.718 & -0.053 \\ -0.998 & -2.412 & 4.360 & -3.740 \end{bmatrix}$$

$J^* = 133.7$

For $n_c = 1$,

$$F^* = \begin{bmatrix} -2.193 & -0.584 & 3.109 & -0.031 & 1.456 \\ -0.934 & -2.458 & 4.808 & -3.643 & 0.642 \\ -1.418 & -0.204 & 3.712 & 0.749 & 1.554 \end{bmatrix}$$

$J^* = 129.624$

For $n_c = 2$,

$$F^* = \begin{bmatrix} -2.592 & -0.3861 & 2.712 & -0.038 & 0.0554 & 0.014 \\ -0.9536 & -2.3890 & 4.282 & -3.707 & 0.0201 & 0.004 \\ -0.0395 & -0.0127 & 0.0852 & 0.008 & 0.0556 & -0.0002 \\ -0.0052 & -0.0056 & 0.0120 & -0.0014 & -0.0001 & 0.5566 \end{bmatrix}$$

$J^* = 136.068.$

TABLE 11 Cost matrices for various X_o's in Example 8

$F = 0$ $$S_o = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}$$

1)

1) $X_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $$S^* = \begin{bmatrix} 3.08 & -1.76 \\ -1.76 & 5.40 \end{bmatrix}$$

2) $X_o = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $$S^* = \begin{bmatrix} 3.08 & -1.76 \\ -1.76 & 5.40 \end{bmatrix}$$

3) $X_o = \begin{bmatrix} 15 & 0 \\ 0 & 1 \end{bmatrix}$ $$S^* = \begin{bmatrix} 1.263 & -0.395 \\ -0.395 & 9.947 \end{bmatrix}$$

4) $X_o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $$S^* = \begin{bmatrix} 1.236 & -3.59E-04 \\ -3.59E-04 & 10.76 \end{bmatrix}$$

CHAPTER 7

CONCLUSION

The algorithm presented in this thesis can be used as a general purpose algorithm for the design of optimal output feedback regulators. The descent Anderson Moore method which has been coded in Fortran IV for creating CAD package performed satisfactorily for a wide class of systems. There are examples for which the method has failed to converge satisfactorily; one of them being the test example given in [28].

The algorithm is written for systems with direct feed-through and for general quadratic performance indices. As stated in Remark 3.3.6, the algorithm can be used for calculating optimal output feedback gains for stochastic systems with plant disturbances assumed as white noise processes.

When faced with convergence problems, penalty on feedback elements could help in convergence and uniqueness of the solution [35]. When the number of free parameters in the feedback gain matrix is large, constraining some of the elements helps in convergence of the algorithm.

The efficiency of the algorithm depends greatly on the design parameters chosen.

APPENDIX

In the system equations,

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x} + D\underline{u}$$

$$\underline{y}_p = C_p\underline{x} + D_p\underline{u}$$

and index of performance

$$J = E\left[\int_0^\infty (\underline{y}_p' \, Q \, \underline{y}_p + \underline{u}' \, R\underline{u}) \, dt\right]$$

with $E(\underline{x}_o \, \underline{x}_o') = X_o$.

A, B, C, D, C_p, D_p, Q, R and X_o for different examples a
given by :

Example 1

$$A = \begin{bmatrix} -0.2 & 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 1.6 & 0 & 0 \\ 0 & 0 & -14.29 & 85.72 & 0 \\ 0 & 0 & 0 & -25 & 75 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix} \qquad C_p' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad D = 0 = D_p$$

$Q = 1$, $R = 1$

Example 2

$$A = \begin{bmatrix} -0.07 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.4 \end{bmatrix} \quad B = \begin{bmatrix} 0.1 & 0 \\ 1 & 0 \\ 0.2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0.4 & 3 \end{bmatrix}$$

$$C_p = I_{3x3} = D = D_p = 0, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 20 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 3

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C' = \begin{bmatrix} 0 & -1 \\ 5 & -1 \\ -1 & 0 \end{bmatrix} \quad C_p = I_{3x3}$$

$$D = D_p = 0, \quad Q = I_{3x3}, \quad R = 1, \quad X_o = I_{3x3}$$

Example 4

$$A = \begin{bmatrix} -0.037 & 0.0123 & 0.00055 & -1 \\ 0 & 0 & 1 & 0 \\ -6.37 & 0 & -0.23 & 0.0618 \\ 1.25 & 0 & 0.016 & -0.0457 \end{bmatrix} \quad B = \begin{bmatrix} 0.00084 & 0.00023 \\ 0 & 0 \\ 0.08 & 0.804 \\ -0.0862 & -0.0665 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_p = I_{4x4}, \; D = D_p = 0,$$

$$Q = I_{4x4}, \; R = I_{2x2}, \; X_o = I_{4x4}$$

Example 5

$$A = \begin{bmatrix} -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.744 & -0.032 & 0 & -0.154 & -0.0042 & 1.54 & 0 \\ 0.337 & -1.12 & 0 & 0.249 & -1.0 & -5.20 & 0 \\ 0.02 & 0 & 0.0386 & -0.996 & -0.00029 & -0.117 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & -0.5 \end{bmatrix}$$

$$B' = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_p = I_{7x7}, \; D_p = D = 0$$

Q = diag. $[1,1,30,30,5,5,1]$, R = diag. $[1,1]$, $X_o = I_{7x7}$

Example 6

$$A = \begin{bmatrix} -0.154 & 0.004 & 0.99 & 0.178 & 0.075 \\ -1.25 & -2.85 & 1.43 & 0 & -0.727 \\ 0.568 & -0.277 & -0.284 & 0 & -2.050 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_p = I_{5x5}, \; D_p = D = 0$$

Example 7

$$A=\begin{bmatrix} 0 & 1 & 0.5 & 1 & 0.6 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \\ 0.5 & 1 & 0 & 2 & 1 & 0.5 \\ 0 & 0.5 & 1 & 3 & 0 & -0.5 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0.5 & 0.5 & 0 & -3 & -4 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$C = C_p = I_{6x6}$, $D_p = D = 0$, $Q = I_{6x6}$, $R = I_{4x4}$, $X_o = I_{6x6}$

Example 8

$A = -I_{2x2}$, $B = I_{2x2}$, $C = [1 \quad 1]$, $C_p = I_{2x2}$, $D = D_p = 0$

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}, \qquad R = I_{2x2}$$

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